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On the Chebyshev Norm of Polynomial B-Splines

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Polynomial B-splines of given order m and with knots of arbitrary multiplicity are investigated with respect to their Chebyshev norm. We present a complete characterization of those B-splines with maximal and those with minimal norm, compute these norms explicitly, and study their behavior as m tends to infinity. Furthermore, the norm of the B-spline corresponding to the equidistant distribution of knots is studied. Moreover, we investigate the behavior of the B-spline's maximum, if a new knot is inserted and/or if one of the knots is moved. Finally, we analyse those types of knot distributions for which the norms of the corresponding B-splines converge to zero as $m \to \infty$. © 1995 Academic Press, Inc.

1. INTRODUCTION

Let $m, k \in \mathbb{N}$ with $m \geqslant k \geqslant 1$. By B_m or, in greater detail, by

$$B_m\left(x \,\middle|\, \begin{matrix} x_0 & x_1 & \cdots\cdots & x_k \\ \nu_0 & \nu_1 & \cdots\cdots & \nu_k \end{matrix}\right)$$

we denote the unique B-spline of order m, belonging to the knots

$$x_0 < x_1 < \cdots < x_k,$$

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where to each knot x_ϱ there is associated a *multiplicity* ν_ϱ, such that

$$\nu_0 + \nu_1 + \cdots + \nu_k = m + 1.$$

Throughout this paper, we choose the normalization

$$\int_{-\infty}^{+\infty} B_m(x)\,dx = \frac{1}{m}.$$

It is well-known (cf. [4, 6, 14]) that $B_m(x) > 0$ for $x \in (x_0, x_k)$. Hence the supremum of B_m on $[x_0, x_k]$ is identical with the value of its Chebyshev norm $\|B_m\|$.

We now consider, for fixed order m, the set $\mathscr{B}_m$ of all B-splines of order m with the normalization $x_0 = 0$ and $x_k = 1$. In this paper we investigate the numbers

$$\lambda_m := \sup\{\|B_m\| \mid B_m \in \mathscr{B}_m\}$$

and

$$\mu_m := \inf\{\|B_m\| \mid B_m \in \mathscr{B}_m\}.$$

We call a B-spline $B_m \in \mathscr{B}_m$ *maximal*, if its norm is equal to λ_m, and *minimal*, it it equals μ_m.

Due to our normalization, each $B_m \in \mathscr{B}_m$ is part of a nonnegative partition of unity, and therefore we have at once the result that $\lambda_m \leqslant 1$ for all m. On the other hand it is known that the norm of each B-spline having a knot of multiplicity m in x_0 or x_1 equals 1, hence we obtain $\lambda_m = 1$. Therefore, we are from now on mainly interested in the numbers μ_m and the corresponding minimal B-splines.

In Section 4 we will compute these numbers explicitly and present all B-splines of $\mathscr{B}_m$ with norm λ_m resp. with norm μ_m. Likewise, in Section 5 the norm of the B-spline with equidistant knots will be computed and the behavior of all these norms as m tends to infinity will be studied. Moreover, in Section 6 we will investigate the dependence of the B-spline's maximum on the insertion of a new knot. The final section 7 is devoted to the question, for which types of knot distributions the norms of the corresponding B-splines tend to zero, as $m \to \infty$.

Before that, in the next sections we give some results on B-splines with a small number of knots, and a contour integral representation for B-splines and their derivatives, which will be our essential tool in proving the results. Some of the results in this paper can be found—with alternative proofs—in the unpublished report [11].

2. B-Splines for a Small Number of Knots

In this section we give for all $m \in \mathbb{N}$ and $k \leqslant 2$ explicit representations for the corresponding B-splines as well as for their norms. The main part of this short section is concerned with basic properties of S. N. Bernstein's basic polynomials, which coincide with the B-splines having no inner knots. Some of them can possibly be traced back to older books on these topic. However, in order to be complete we prefer to present proofs for almost all these elementary results, which will later turn out to be important.

We start with the observation that for $m \in \mathbb{N}$ and $\nu_0 + \nu_1 = m+1$ we have

$$B_m\left(x \,\middle|\, \begin{matrix} 0 & 1 \\ \nu_0 & \nu_1 \end{matrix}\right) = \begin{cases} \binom{m-1}{\nu_0 - 1} x^{m-\nu_0}(1-x)^{m-\nu_1} & \text{for } 0 \leqslant x \leqslant 1, \\ 0 & \text{elsewhere}, \end{cases} \tag{2.1}$$

and

$$\left\| B_m\left(\cdot \,\middle|\, \begin{matrix} 0 & 1 \\ \nu_0 & \nu_1 \end{matrix}\right)\right\| = \binom{m-1}{\nu_0 - 1} \frac{(m-\nu_0)^{m-\nu_0}(m-\nu_1)^{m-\nu_1}}{(m-1)^{m-1}}, \tag{2.2}$$

which can easily be checked. This enables us to prove the following lemma.

Lemma 1. *Let α_m denote the minimal norm of all B-splines from $\mathscr{B}_m$ with no inner knots, i.e.,*

$$\alpha_m := \min\left\{ \left\| B_m\left(\cdot \,\middle|\, \begin{matrix} 0 & 1 \\ \nu_0 & \nu_1 \end{matrix}\right)\right\| \,\middle|\, \nu_0 + \nu_1 = m+1 \right\}.$$

Then

$$\alpha_m = \begin{cases} \dfrac{1}{2^{m-1}} \dbinom{m-1}{(m-1)/2} & \text{if } m \text{ is odd}, \\ \dfrac{1}{2^{m-1}} \dbinom{m-1}{m/2} \dfrac{m^{m/2}(m-2)^{(m-2)/2}}{(m-1)^{m-1}} & \text{if } m \text{ is even}. \end{cases} \tag{2.3}$$

Furthermore, one has

$$\alpha_m > \alpha_{m+1} \quad \text{for} \quad m \geqslant 2. \tag{2.4}$$

Proof. It follows directly from Equation (2.2) that

$$\left\| B_1\left(\cdot \,\middle|\, \begin{matrix} 0 & 1 \\ \nu_1 & \nu_2 \end{matrix}\right)\right\| = \left\| B_2\left(\cdot \,\middle|\, \begin{matrix} 0 & 1 \\ \nu_1 & \nu_2 \end{matrix}\right)\right\| = 1$$

for all possible choices of ν_1 and ν_2. Hence formula (2.3) is true for $m = 1, 2$, so we may assume from now on that $m \geqslant 3$.

Let ν_0 increase from 1 to the largest integer less than or equal to $(m+1)/2$, and replace ν_1 in (2.2) always by $m+1-\nu_0$. We investigate the question for which values of ν_0 the expression

$$\Phi(\nu_0) := \left\| B_m\left(\cdot \,\middle|\, \begin{matrix} 0 & 1 \\ \nu_0 & m+1-\nu_0 \end{matrix} \right) \right\|$$

is minimal. Obviously $\Phi(1) = 1$. For $1 < \nu_0 \leqslant (m+1)/2$ we will prove the inequality

$$\Phi(\nu_0 + 1) < \Phi(\nu_0). \tag{2.5}$$

Using the expression given in (2.2), it can easily be concluded that the inequality (2.5) is equivalent to

$$\left(\frac{\nu_0}{\nu_0 - 1} \right)^{\nu_0 - 1} < \left(\frac{m - \nu_0}{m - \nu_0 - 1} \right)^{m - \nu_0 - 1}. \tag{2.6}$$

Now the function

$$g(t) := \left(\frac{t}{t-1} \right)^{t-1}, \qquad t \in \mathbb{R}, \quad t \geqslant 2,$$

is strictly increasing. Hence the inequality (2.6) is valid if and only if $\nu_0 < m - \nu_0$, i.e., $\nu_0 < m/2$ holds. It follows that the minimal value in question is attained for $\nu_0 = \nu_1 = (m+1)/2$ if m is odd. If m is even, then the minimal value is attained for $\nu_0 = m/2$, $\nu_1 = (m+2)/2$, and, by means of symmetry, also for $\nu_0 = (m+2)/2$, $\nu_1 = m/2$, and no other cases. Formula (2.3) now follows immediately.

Finally we have to prove the inequality (2.4). It is easily verified that

$$\alpha_2 = 1 > \alpha_3 = \frac{1}{2} > \alpha_4 = \frac{4}{9}.$$

For odd m, say $m = 2r+1$ with $r \geqslant 1$, we get from (2.3) at once

$$\alpha_m - \alpha_{m+1} = \frac{1}{2^{2r}} \binom{2r}{r} \left\{ 1 - \left(\frac{4r(r+1)}{4r(r+1)+1} \right)^r \right\} > 0.$$

For even m, say $m = 2r+2$ with $r \geqslant 1$, we obtain

$$\alpha_m - \alpha_{m+1} = \frac{1}{2^{2r+2}} \binom{2r+2}{r+1} \left\{ \frac{2r+2}{2r+1} \left(\frac{4r(r+1)}{4r(r+1)+1} \right)^r - 1 \right\} > 0,$$

since

$$\frac{2r+2}{2r+1}\left(\frac{4r(r+1)}{4r(r+1)+1}\right)^r$$
$$> \frac{2r+2}{2r+1}\left(1-\frac{r}{4r(r+1)+1}\right)=\frac{8r^3+14r^2+8r+2}{8r^3+12r^2+6r+1}>1. \quad \blacksquare$$

We finish this section by giving two examples of B-splines and their corresponding norms, which will be needed later on.

LEMMA 2. *For $m \geqslant 2$ we have*

$$B_m\left(x \,\middle|\, \begin{matrix} 1 & x_1 & 1 \\ 0 & m-1 & 1 \end{matrix}\right) = \begin{cases} \left(\dfrac{x}{x_1}\right)^{m-1} & \text{for } 0 \leqslant x < x_1, \\ \left(\dfrac{1-x}{1-x_1}\right)^{m-1} & \text{for } x_1 \leqslant x \leqslant 1, \\ 0 & \text{elsewhere}, \end{cases}$$

and

$$B_m\left(x \,\middle|\, \begin{matrix} 1 & x_1 & 1 \\ 0 & 1 & m-1 \end{matrix}\right) = \begin{cases} \dfrac{x^{m-1}}{x_1} & \text{for } 0 \leqslant x < x_1, \\ \dfrac{1}{x_1}\cdot\left(x^{m-1}-\left(\dfrac{x-x_1}{1-x_1}\right)^{m-1}\right) & \text{for } x_1 \leqslant x \leqslant 1, \\ 0 & \text{elsewhere}. \end{cases}$$

Furthermore

$$\left\| B_m\left(\cdot \,\middle|\, \begin{matrix} 0 & x_1 & 1 \\ 1 & m-1 & 1 \end{matrix}\right)\right\| = 1 \quad \text{and}$$
$$\left\| B_m\left(\cdot \,\middle|\, \begin{matrix} 0 & x_1 & 1 \\ 1 & 1 & m-1 \end{matrix}\right)\right\| = \left(\frac{x_1}{1-(1-x_1)^{m-1/(m-2)}}\right)^{m-2}.$$

Proof. These properties are direct consequences of the definition of B_m. $\blacksquare$

3. CONTOUR INTEGRAL REPRESENTATIONS OF B-SPLINES AND THEIR PARTIAL DERIVATIVES

In this section, we present some results concerning the representation of the B-spline B_m and of its partial derivatives with respect to the knots in terms of a complex contour integral. These results will also be a major tool for the proof of our Theorem 2. For convenience, we first repeat the well-known contour integral representation of the B-spline itself:

LEMMA 3. *Let, for $x \in \mathbb{R}$, C_x denote a simply closed and rectifiable curve in the complex plane, such that all the knots x_ϱ, $\varrho = 0, ..., k$, with $x < x_\varrho$ and no others lie in the interior of that curve.*

Then, carrying out the integration in the positive sense, we have the representation

$$B_m\left(x \,\middle|\, \begin{matrix} 0 & x_1 & \cdots & x_{k-1} & 1 \\ \nu_0 & \nu_1 & \cdots & \nu_{k-1} & \nu_k \end{matrix}\right) = \frac{1}{2\pi i} \int_{C_x} \frac{(z-x)^{m-1}}{\omega(x)}\, dz, \tag{3.1}$$

where

$$\omega(z) := z^{\nu_0}(z-x_1)^{\nu_1} \cdots (z-x_{k-1})^{\nu_{k-1}}(z-1)^{\nu_k}. \tag{3.2}$$

Proof. This result was given in [9], see also [5]. ∎

In our subsequent considerations, representation (3.1) will mainly serve as a theoretical tool. However, it should be emphasized that this formula has also important practical implications, a fact which seems to have been underestimated until now, although (3.1) is known since twenty years at least[1]. Therefore we would like to make a few remarks on this subject first:

COROLLARY. *Let $x \in (x_{\varrho-1}, x_\varrho)$ for some ϱ, and put for all μ*

$$\omega_\mu(z) := (z-x_\mu)^{-\nu_\mu} \cdot \omega(z)$$

with ω defined in (3.2). Then the following representation holds:

$$B_m\left(x \,\middle|\, \begin{matrix} 0 & x_1 & \cdots & x_{k-1} & 1 \\ \nu_0 & \nu_1 & \cdots & \nu_{k-1} & \nu_k \end{matrix}\right) = \sum_{\mu=0}^{\varrho-1} \frac{(-1)^m}{(\nu_\mu - 1)!} \cdot \frac{d^{\nu_\mu - 1}}{dz^{\nu_\mu - 1}} \left(\frac{(x-z)^{m-1}}{\omega_\mu(z)}\right)_{z=x_\mu} \tag{3.3}$$

Proof. According to the residue theorem, we obtain from (3.1)

$$\begin{aligned} B_m\left(x \,\middle|\, \begin{matrix} 0 & x_1 & \cdots & x_{k-1} & 1 \\ \nu_0 & \nu_1 & \cdots & \nu_{k-1} & \nu_k \end{matrix}\right) &= \sum_{\mu=\varrho}^{k} \operatorname{Res}_{z=x_\mu}\left(\frac{(z-x)^{m-1}}{\omega(z)}\right) \\ &= -\sum_{\mu=0}^{\varrho-1} \operatorname{Res}_{z=x_\mu}\left(\frac{(z-x)^{m-1}}{\omega(z)}\right). \end{aligned}$$

[1] We thank the referee for pointing out to us that formula (3.1) was found indepedently by Tschakaloff [15] already in 1938; for more information on this topic, see [2, p. 39].

Since

$$\operatorname{Res}_{z=x_\mu}\left(\frac{(z-x)^{m-1}}{\omega(z)}\right)=\frac{1}{(\nu_\mu-1)!}\cdot\frac{d^{\nu_\mu-1}}{dz^{\nu_\mu-1}}\left(\frac{(z-x)^{m-1}}{\omega_\mu(z)}\right)_{z=x_\mu}$$

(see any textbook on complex analysis), the result follows. ∎

If we carry out the differentiation in (3.3) explicitly, we see that our B-spline B_m is of the form

$$B_m\left(x\,\middle|\,\begin{matrix}0 & x_1 & \cdots & x_{k-1} & 1\\ \nu_0 & \nu_1 & \cdots & \nu_{k-1} & \nu_k\end{matrix}\right)=\sum_{\mu=0}^{k}\sum_{j=1}^{\nu_\mu}\beta_{\mu j}(x-x_\mu)_+^{m-j} \tag{3.4}$$

with

$$\beta_{\mu\nu_\mu}\neq 0 \qquad \text{for all } \mu, \tag{3.5}$$

which so far is a known result, see e.g. [14, Theorem 4.14]. But in contrast to the usual divided-difference approach, the calculation of the $\beta'_{\mu j}s$ via eqn. (3.3) is—for concrete cases—not very difficult to do. For example, we easily recognize that for all μ

$$\beta_{\mu\nu_\mu}=\binom{m-1}{\nu_\mu-1}\frac{(-1)^{m+\nu_\mu-1}}{\omega_\mu(x_\mu)}, \tag{3.6}$$

which sharpens assertion (3.5).

THEOREM 1. *Let $k\in\mathbb{N}$, $k\geqslant 2$. The multiplicities ν_ϱ of the knots x_ϱ, $\varrho=1, 2, ..., k-1$ are all assumed to satisfy $\nu_\varrho<m-1$. Then the B-spline*

$$B_m\left(x\,\middle|\,\begin{matrix}0 & x_1 & \cdots & x_{k-1} & 1\\ \nu_0 & \nu_1 & \cdots & \nu_{k-1} & \nu_k\end{matrix}\right)$$

possesses continuous partial derivatives with respect to the variable x and to all the knots $x_1, x_2, ..., x_{k-1}$ in the Cartesian product $(0,1)\times D$, where

$$D:=\{x_1, x_2, ..., x_{k-1}\mid 0<x_1<\cdots<x_{k-1}<1\}.$$

Furthermore we have the representations

$$\frac{\partial}{\partial x}B_m\left(x\,\middle|\,\begin{matrix}0 & x_1 & \cdots & x_{k-1} & 1\\ \nu_0 & \nu_1 & \cdots & \nu_{k-1} & \nu_k\end{matrix}\right)=-\frac{m-1}{2\pi i}\int_{C_x}\frac{(z-x)^{m-2}}{\omega(z)}\,dz, \tag{3.7}$$

and, for $\varrho=1, 2, ..., k-1$,

$$\frac{\partial}{\partial x_\varrho}B_m\left(x\,\middle|\,\begin{matrix}0 & x_1 & \cdots & x_{k-1} & 1\\ \nu_0 & \nu_1 & \cdots & \nu_{k-1} & \nu_k\end{matrix}\right)=\frac{\nu_\varrho}{2\pi i}\int_{C_x}\frac{(z-x)^{m-1}}{\omega(x)(z-x_\varrho)}\,dz. \tag{3.8}$$

Proof. The differentiability with respect to x follows from the assumption $\nu_r < m-1$, $\varrho = 1, ..., k-1$. Since C_x is rectifiable, formula (3.7) is easily derived.

The right hand side of (3.8) possesses a denominator, in which the multiplicity of each knot is still less than m. Hence this right hand side is continuous in the Cartesian product $(0, 1) \times D$. It obviously represents the partial derivative of B_m with respect to the knot x_ϱ. ∎

As an immediate consequence of Theorem 1 we have therefore a new proof for the well-known relation (cf. [14])

$$\frac{\partial}{\partial x_\varrho} B_m\left(x \,\middle|\, \begin{matrix} 0 & x_1 & \cdots & x_{k-1} & 1 \\ \nu_0 & \nu_1 & \cdots & \nu_{k-1} & \nu_k \end{matrix}\right)$$
$$= -\frac{\nu_\varrho}{m} \cdot B'_{m+1}\left(x \,\middle|\, \begin{matrix} 0 & x_1 & \cdots & x_\varrho & \cdots & x_{k-1} & 1 \\ \nu_0 & \nu_1 & \cdots & \nu_\varrho + 1 & \cdots & \nu_{k-1} & \nu_k \end{matrix}\right),$$

where B'_{m+1} denotes the derivative of B_{m+1} with respect to the variable x.

4. B-Splines with Largest and Smallest Chebyshev Norm

In this section we want to compute explicitly the numbers μ_m, defined in the introduction. Let us first consider the elementary cases $m = 1, 2, 3$, where the last one soon will turn out to be typical also for the general case.

For $m = 1$ and $m = 2$ it follows immediately from Lemma 2 that

$$\mu_1 = \mu_2 = 1.$$

Also the case $m = 3$ can still be treated in an elementary way. For $0 < x_1 < x_2 < 1$ we have

$$B_3\left(x \,\middle|\, \begin{matrix} 0 & x_1 & x_2 & 1 \\ 1 & 1 & 1 & 1 \end{matrix}\right)$$
$$= \begin{cases} \dfrac{x^2}{x_1 x_2} & \text{for } 0 \leqslant x < x_1, \\[2mm] \dfrac{-x^2(1 + x_2 - x_1) + 2x x_2 - x_1 x_2}{(1 - x_1)\, x_2 (x_2 - x_1)} & \text{for } x_1 \leqslant x < x_2, \\[2mm] \dfrac{(1-x)^2}{(1-x_1)(1-x_2)} & \text{for } x_2 \leqslant x \leqslant 1, \\[2mm] 0 & \text{elsewhere.} \end{cases}$$

The maximum value of this B-spline is located at the point

$$\tau := \frac{x_2}{1 + x_2 - x_1},$$

and the norm turns out to be

$$\left\| B_3\left(\cdot \,\middle|\, \begin{matrix} 0 & x_1 & x_2 & 1 \\ 1 & 1 & 1 & 1 \end{matrix}\right) \right\| = \frac{1}{1 + x_2 - x_1},$$

hence $\mu_3 \leqslant 1/2$. The remaining cases of double and triple knots are analysed in Section 2. Here it is also possible to consider the limits for $x_1 \to 0$ or $x_2 \to 1$ or $x_2 \to x_1$ etc. It turns out that the maximum value $\lambda_3 = 1$ of the norm is only attained by the B-splines

$$B_3\left(x \,\middle|\, \begin{matrix} 0 & 1 \\ 1 & 3 \end{matrix}\right), \quad B_3\left(x \,\middle|\, \begin{matrix} 0 & 1 \\ 3 & 1 \end{matrix}\right) \quad \text{and} \quad B_3\left(x \,\middle|\, \begin{matrix} 0 & x_1 & 1 \\ 1 & 2 & 1 \end{matrix}\right) \; (0 < x_1 < 1).$$

Analogously we find that $\mu_3 = 1/2$, where this value is attained only by the B-spline

$$B_3\left(x \,\middle|\, \begin{matrix} 0 & 1 \\ 2 & 2 \end{matrix}\right) = \begin{cases} 2x(1-x) & \text{for } 0 \leqslant x \leqslant 1, \\ 0 & \text{elsewhere.} \end{cases}$$

We now give a complete description of the situation for arbitrary order m.

THEOREM 2. *Let m be any natural number. Then the following assertions hold*:

1. *We have*

$$\mu_m = \alpha_m,$$

where α_m is given in Lemma 1.

2. *The maximum value $\lambda_m = 1$ of the norm is attained by the B-splines*

$$B_m\left(x \,\middle|\, \begin{matrix} 0 & 1 \\ 1 & m \end{matrix}\right), \quad B_m\left(x \,\middle|\, \begin{matrix} 0 & 1 \\ m & 1 \end{matrix}\right)$$

and

$$B_m\left(x \,\middle|\, \begin{matrix} 0 & x_1 & 1 \\ 1 & m-1 & 1 \end{matrix}\right) \; (0 < x_1 < 1, \; m \geqslant 2), \tag{4.1}$$

and by no others.

The minimum value μ_m of the norm is attained by the B-splines

$$B_m\left(x \,\middle|\, \begin{matrix} 0 & 1 \\ \frac{m+1}{2} & \frac{m+1}{2} \end{matrix}\right) \quad (m \text{ odd}) \tag{4.2}$$

and

$$B_m\left(x \,\middle|\, \begin{matrix} 0 & 1 \\ \frac{m}{2} & \frac{m+2}{2} \end{matrix}\right) \quad \text{and} \quad B_m\left(x \,\middle|\, \begin{matrix} 0 & 1 \\ \frac{m+2}{2} & \frac{m}{2} \end{matrix}\right) \quad (m \text{ even}) \tag{4.3}$$

and by no others.

Proof. Obviously, we only have to prove the second assertion, since the first one then easily follows from the results in Section 2.

We proceed by induction with respect to m and note that the cases $m = 1, 2, 3$ have already been proved. Hence assume $m \geqslant 4$. Let $k \in \mathbb{N}$, $k \geqslant 2$, and consider any fixed B-spline

$$B_m\left(x \,\middle|\, \begin{matrix} 0 & x_1 & \cdots & x_{k-1} & 1 \\ v_0 & v_1 & \cdots & v_{k-1} & v_k \end{matrix}\right).$$

We claim that under the assumptions of Theorem 1, the gradient vector

$$\operatorname{grad} B_m = \left\{\frac{\partial B_m}{\partial x}, \frac{\partial B_m}{\partial x_1}, \ldots, \frac{\partial B_m}{\partial x_{k-1}}\right\} \tag{4.4}$$

never vanishes on the Cartesian product $(0, 1) \times D$. If $k = 2$, we assume in addition that $v_1 < m - 2$. The remaining case $k = 2$, $v_1 = m - 2$ will be treated separately. Note that our claim implies that all the corresponding B-splines are neither maximal nor minimal. In order to prove it we assume to the contrary that there is a point

$$p := (\tau, y_1, \ldots, y_{k-1}) \in (0, 1) \times D$$

such that simultaneously

$$\left(\frac{\partial B_m}{\partial x}\right)_p = 0 \quad \text{and} \quad \left(\frac{\partial B_m}{\partial x_\varrho}\right)_p = 0 \quad \text{for } \varrho = 1, 2, \ldots, k-1$$

hold. According to (3.7) and (3.8), the representations

$$\left(\frac{\partial B_m}{\partial x}\right)_p = -\frac{m-1}{2\pi i}\int_{C_\tau}\frac{(z-\tau)^{m-1}}{\omega(z)(z-\tau)}\,dz$$

and, for $\varrho = 1, 2, ..., k-1$,

$$\left(\frac{\partial B_m}{\partial x_\varrho}\right)_p = \frac{\nu_\varrho}{2\pi i}\int_{C_\tau}\frac{(z-\tau)^{m-1}}{\omega(z)(z-y_\varrho)}\,dz$$

are valid. Here, of course

$$\omega(z) = z^{\nu_0}(z-y_1)^{\nu_1}\cdots(z-y_{k-1})^{\nu_{k-1}}(z-1)^{\nu_k}.$$

Our assumption yields that, if τ does not coincide with one of the knots $y_1, ..., y_{k-1}$, the linear functional L_1, defined on the space of entire functions h by

$$L_1 h := \frac{1}{2\pi i}\int_{C_\tau}\frac{(z-\tau)^{m-2}h(z)}{\omega^*(z)}\,dz \tag{4.5}$$

where

$$\omega^*(z) := z^{\nu_0}(z-y_1)^{\nu_1+1}\cdots(z-y_{k-1})^{\nu_{k-1}+1}(z-1)^{\nu_k},$$

vanishes for all $h \in \Pi_{k-1}$. If τ coincides with one of these inner knots, then the linear functional L_2, defined on the space of entire functions h by

$$L_2 h := \frac{1}{2\pi i}\int_{C_\tau}\frac{(z-\tau)^{m-1}h(z)}{\omega^*(z)}\,dz$$

vanishes for all $h \in \Pi_{k-2}$. In order to save certain differentiability properties we assume in the second case $k > 2$. The remaining case $k = 2$, $\tau = y_1$ again will be treated separately.

Let us consider the first case now. We choose the entire function $h_1(z) := (z-\tau)^{k-1}$; then

$$\begin{aligned} L_1 h_1 &= \frac{1}{2\pi i}\int_{C_\tau}\frac{(z-\tau)^{m+k-3}}{\omega^*(z)}\,dz \\ &= \frac{-1}{m+k-2}\left(\frac{\partial}{\partial x}B_{m+k-1}\left(x \,\middle|\, \begin{matrix} 0 & y_1 & \cdots & y_{k-1} & 1 \\ \nu_0 & \nu_1+1 & \cdots & \nu_{k-1}+1 & \nu_k \end{matrix}\right)\right)_{x=\tau} = 0. \end{aligned} \tag{4.6}$$

Since it is well-known (see [14, Thm. 4.57]) that the first derivative of this B-spline B_{m+k-1} can have only one root in $(0, 1)$, the number

τ is uniquely determined by equation (4.6). Next we choose $h_2(z) := (z-\tau)^{k-2}$; then

$$L_1 h_2 = \frac{1}{2\pi i} \int_{C_\tau} \frac{(z-\tau)^{m+k-4}}{\omega^*(z)}\, dz$$
$$= \frac{1}{(m+k-2)(m+k-3)}$$
$$\times \left(\frac{\partial^2}{\partial x^2} B_{m+k-1} \left(x \,\middle|\, \begin{matrix} 0 & y_1 & \cdots & y_{k-1} & 1 \\ \nu_0 & \nu_1+1 & \cdots & \nu_{k-1}+1 & \nu_k \end{matrix} \right) \right)_{x=\tau}$$
$$= 0.$$

But since the first derivative of our B-spline B_{m+k-1} already vanishes at $x=\tau$, this cannot hold also for the second derivative, see again [14]. Therefore our assumption leads to a contradiction, and the original assertion is proved in the first case.

In the second case, i.e., if τ coincides with one of the knots $y_1, ..., y_{k-1}$, we choose $h_3(z) := (z-\tau)^{k-2}\,(=h_2(z))$ and $h_4(z) := (z-\tau)^{k-3}$ to obtain

$$L_2 h_3 = 0 \qquad \text{and} \qquad L_2 h_4 = 0,$$

which again, up to non-zero factors, can be interpreted as

$$\left(\frac{\partial}{\partial x} B_{m+k-1} \right)_{x=\tau} = 0 \qquad \text{and} \qquad \left(\frac{\partial^2}{\partial x^2} B_{m+k-1} \right)_{x=\tau} = 0$$

for the same B-spline as before. We get the same contradiction.

Now let us consider the case $k=2$, $\tau = y_1$. Here we have

$$\frac{1}{2\pi i} \int_{C_\tau} \frac{(z-\tau)^{m-\nu_1-2}}{z^{\nu_0}(z-1)^{\nu_2}}\, dz = 0,$$

where $\nu_0 + \nu_2 = m + 1 - \nu_1$. Hence, having $\tau = y_1$ in mind, one has

$$B_m \left(\tau \,\middle|\, \begin{matrix} 0 & y_1 & 1 \\ \nu_0 & \nu_1 & \nu_2 \end{matrix} \right) = \frac{1}{2\pi i} \int_{C_\tau} \frac{(z-\tau)^{m-\nu_1-1}}{z^{\nu_0}(z-1)^{\nu_2}}\, dz$$
$$= B_{m-\nu_1} \left(\tau \,\middle|\, \begin{matrix} 0 & 1 \\ \nu_0 & \nu_2 \end{matrix} \right)$$

and thus

$$\left\| B_m \left(\cdot \,\middle|\, \begin{matrix} 0 & y_1 & 1 \\ \nu_0 & \nu_1 & \nu_2 \end{matrix} \right) \right\| = \left\| B_{m-\nu_1} \left(\cdot \,\middle|\, \begin{matrix} 0 & 1 \\ \nu_0 & \nu_2 \end{matrix} \right) \right\|$$
$$\geqslant \mu_{m-\nu_1} = \alpha_{m-\nu_1} > \alpha_m \geqslant \mu_m,$$

where we have used inequality (2.4) and the induction hypothesis.

There remain the cases $k=2$, $\nu_1=m-2$, i.e., we still have to show that the B-splines

$$B_m\left(x\,\middle|\,\begin{matrix}0 & x_1 & 1\\ 1 & m-2 & 2\end{matrix}\right) \quad\text{and}\quad B_m\left(x\,\middle|\,\begin{matrix}0 & x_1 & 1\\ 2 & m-2 & 1\end{matrix}\right)$$

are neither maximal nor minimal. Due to symmetry reasons, we may restrict to the first one. It can easily be verified that

$$B_m\left(x\,\middle|\,\begin{matrix}0 & x_1 & 1\\ 1 & m-2 & 2\end{matrix}\right) = \begin{cases} \dfrac{x^{m-1}}{x_1^{m-2}} & \text{for} \quad 0\leqslant x<x_1,\\[2ex] \dfrac{(m-1)(1-x)^{m-2}-\left(1+\dfrac{m-2}{1-x_1}\right)(1-x)^{m-1}}{(1-x_1)^{m-2}} & \text{for} \quad x_1\leqslant x\leqslant 1,\\[2ex] 0 & \text{elsewhere.}\end{cases}$$

The maximum value of B_m occurs at the point τ with

$$1-\tau=\frac{(m-2)(1-x_1)}{m-x_1-1},$$

and it follows

$$\left\|B_m\left(\cdot\,\middle|\,\begin{matrix}0 & x_1 & 1\\ 1 & m-2 & 2\end{matrix}\right)\right\| = \left(\frac{m-2}{m-x_1-1}\right)^{m-2}. \tag{4.7}$$

Obviously, this function is neither maximal nor minimal.

So, the B-splines which are either maximal or minimal have to be of the type (2.1) or (4.1); these cases have been discussed earlier. ∎

Remark. It is also possible to give an alternative proof of Theorem 2, which is mainly based on Theorem 8 (see section 7). This was worked out in [11].

We are now in the position to characterize the asymptotic behaviour of the minimal norms μ_m, as m goes to infinity.

THEOREM 3. *The sequence* $\{\mu_m\}$ *satisfies the asymptotic relation*

$$\mu_m=\sqrt{\frac{2}{\pi m}}\left(1+\frac{1}{4m}+O(m^{-2})\right) \quad \textit{for} \quad m\to\infty. \tag{4.8}$$

Remark. Relation (4.8) implies in particular that

$$\mu_m = O(m^{-1/2}) \qquad \text{for} \quad m \to \infty,$$

which has already been conjectured in [9].

Proof of Theorem 3. Assume first that m is odd, say $m = 2k+1$ with $k \geqslant 1$; then, according to Lemma 1,

$$\mu_{2k+1} = \frac{1}{2^{2k}} \binom{2k}{k}. \tag{4.9}$$

This is nothing else but the famous Wallis product, which is known to possess the asymptotic expansion (see [1, #6.1.49])

$$\frac{1}{2^{2k}} \binom{2k}{k} = \frac{1}{\sqrt{\pi k}} \left(1 - \frac{1}{8k} + O(k^{-2})\right) \qquad \text{for} \quad k \to \infty. \tag{4.10}$$

Now we replace k by $(m-1)/2$ in equation (4.10). This yields

$$\begin{aligned}
\frac{1}{2^{m-1}} \binom{m-1}{(m-1)/2} &= \sqrt{\frac{2}{\pi(m-1)}} \left(1 - \frac{1}{4(m-1)} + O(m^{-2})\right) \\
&= \sqrt{\frac{2}{\pi m}} \left(1 - \frac{1}{m}\right)^{-1/2} \left(1 - \frac{1}{4m} + O(m^{-2})\right) \\
&= \sqrt{\frac{2}{\pi m}} \left(1 + \frac{1}{4m} + O(m^{-2})\right)
\end{aligned}$$

for $m \to \infty$.

Now let m be even, $m = 2k$; Lemma 1 implies

$$\begin{aligned}
\mu_{2k} &= \frac{1}{2^{2k-1}} \binom{2k-1}{k} \frac{(2k)^k (2k-2)^{k-1}}{(2k-1)^{2k-1}} \\
&= \frac{1}{2^{2k}} \binom{2k}{2} \frac{(2k)^k (2k-2)^{k-1}}{(2k-1)^{2k-1}} \\
&= \frac{1}{\sqrt{\pi k}} \left(1 - \frac{1}{8k} + O(k^{-2})\right) \frac{\left(1 - \frac{1}{k}\right)^{k-1}}{\left(1 - \frac{1}{2k}\right)^{2k-1}} \\
&= \frac{1}{\sqrt{\pi k}} \left(1 - \frac{1}{8k} + O(k^{-2})\right)\left(1 + \frac{1}{4k} + O(k^{-2})\right) \\
&= \frac{1}{\sqrt{\pi k}} \left(1 + \frac{1}{8k} + O(k^{-2})\right) \qquad \text{for} \quad k \to \infty.
\end{aligned} \tag{4.11}$$

Here we have used twice the asymptotic relation

$$\left(1-\frac{1}{k}\right)^{k-1}=\exp\left((k-1)\log\left(1-\frac{1}{k}\right)\right)$$
$$=\exp\left(-1+\frac{1}{2k}+O(k^{-2})\right)$$
$$=\exp(-1)\cdot\left(1+\frac{1}{2k}+O(k^{-2})\right).$$

Putting $k=m/2$ in (4.11) yields the assertion. ∎

Obviously, more terms of the asymptotic expansion (4.8) can be worked out easily. The numerical values of the first ten numbers μ_m are given in Table I (see Section 5).

5. The Equidistant Distribution of Knots

In many applications B-splines with *equally spaced* knots are of particular interest. In the context of our investigations, it is therefore natural to ask for the behaviour of their norms; so, let

$$B_m^e(x):=B_m\left(x\,\middle|\,\begin{matrix}0 & \frac{1}{m} & \cdots & \frac{m-1}{m} & 1\\ 1 & 1 & \cdots & 1 & 1\end{matrix}\right)$$

and

$$\beta_m:=\|B_m^e\|.$$

Since $B_m^e(x)=B_m^e(1-x)$ for all x, the norm of this function is attained at $x=1/2$. Hence

$$\beta_m=B_m^e\left(\frac{1}{2}\right)=\frac{1}{(m-1)!\,2^{m-1}}\sum_{\mu=0}^{[(m-1)/2]}(-1)^\mu\binom{m}{\mu}(m-2\mu)^{m-1}, \quad (5.1)$$

where we have used (3.4) and (3.6). In Table I, we list the first ten values of β_m and compare them with the corresponding "optimal" values μ_m; furthermore, we present the asymptotic limits (cf. Thms. 3 and 4).

For $m\to\infty$, we obtain the following asymptotic result:

THEOREM 4. *The sequence of norms of the equidistant B-splines satisfies the asymptotic relation*

$$\beta_m=\sqrt{\frac{6}{\pi m}}\left(1-\frac{3}{20m}+O(m^{-2})\right) \quad \textit{for} \quad m\to\infty. \quad (5.2)$$

TABLE I

m	μ_m	$\sqrt{2/(\pi m)}$	β_m	$\sqrt{6/(\pi m)}$
1	1.00000	0.79788	1.00000	1.38197
2	1.00000	0.56418	1.00000	0.97720
3	0.50000	0.46065	0.75000	0.79788
4	0.44444	0.39384	0.66666	0.69098
5	0.37500	0.35682	0.59895	0.61803
6	0.34560	0.32573	0.55000	0.56418
7	0.31250	0.30157	0.51102	0.52233
8	0.29375	0.28209	0.47936	0.48860
9	0.27343	0.26596	0.45292	0.46065
10	0.26018	0.25231	0.43041	0.43701

Proof. We use Schoenberg's integral representation for cardinal B-splines (cf. [14, Theorem 4.33] or [13, p. 12]), which in our case takes the form

$$B_m^e(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin t}{t}\right)^m e^{imt(2x-1)}\, dt,$$

and so

$$\beta_m = B_m^e\left(\frac{1}{2}\right) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin t}{t}\right)^m dt. \tag{5.3}$$

This integral is treated in several places in the literature (see [8] or [12, p. 94]), and there one can also find the asymptotic expansion (5.2). ∎

Interestingly, the values of β_m (equidistant case) and μ_m (minimal case) both tend to zero with the same order of convergence, and, moreover, the asymptotic constants only differ by a factor $\sqrt{3}$. So, the equidistant knot distribution is, from this paper's point of view, a rather good choice.

6. Some Recurrence Relations Concerning Knot Insertion

In this section we want to study what happens to the location and to the value of a B-spline's maximum, if an additional knot is inserted, and/or if one of the knots is moved. To do this, we need the result of the following Theorem 5.

For $m \geqslant 2$, let there be given a B-spline

$$B_{m-1}(x) = B_{m-1}\left(x \,\middle|\, \begin{matrix} 0 & x_1 & \cdots & x_{k-1} & 1 \\ \nu_0 & \nu_1 & \cdots & \nu_{k-1} & \nu_k \end{matrix}\right)$$

of order $m-1$ (i.e., $\sum_{\varrho=0}^{k} \nu_\varrho = m$). Now we add an arbitrary additional knot $\xi \in (0, 1)$ to the given set of knots (where it does not matter if ξ coincides with one of the x_ϱ's) and obtain a B-spline B_m of order m. Then the following relation holds:

THEOREM 5. *At all points $x \in [0, 1]$, where B'_m exists, one has*

$$B_m(x) = \frac{x-\xi}{m-1} \cdot B'_m(x) + B_{m-1}(x). \tag{6.1}$$

Prof. We multiply the trivial identity

$$\frac{z-x}{z-\xi} = 1 + \frac{\xi - x}{z-\xi}$$

by

$$\frac{(z-x)^{m-2}}{z^{\nu_0}(z-x_1)^{\nu_1} \cdots (z-x_{k-1})^{\nu_{k-1}} (z-1)^{\nu_k}}$$

and integrate over the simply closed path C_x described in (3.1). Then formula (6.1) follows directly from the representation (3.1) in combination with (3.7). ▮

Remark. There exist at least two other completely different proofs of equation (6.1), see [11].

Relation (6.1) can be considered as a differential equation for the function B_m. The solution of this equation, which can easily be obtained, is given in the next lemma.

LEMMA 4. *Using the notation of the previous theorem, the following relation holds*:

$$B_m(x) = \begin{cases} (m-1)(\xi - x)^{m-1} \displaystyle\int_0^x \frac{B_{m-1}(\tau)}{(\xi-\tau)^m}\, d\tau, & 0 \leqslant x < \xi, \quad (6.2\text{a}) \\ B_{m-1}(x), & x = \xi, \quad (6.2\text{b}) \\ (m-1)(x-\xi)^{m-1} \displaystyle\int_x^1 \frac{B_{m-1}(\tau)}{(\tau-\xi)^m}\, d\tau, & \xi < x \leqslant 1. \quad (6.2\text{c}) \end{cases}$$

Moreover, if we assume that B_{m-1} is continuously differentiable in $[0, 1]$, we can show that B'_m can be expressed in terms of B_{m-1} as follows.

LEMMA 5.

$$B'_m(x) = \begin{cases} (m-1)(\xi - x)^{m-2} \displaystyle\int_0^x \frac{dB_{m-1}(\tau)}{(\xi-\tau)^m}, & 0 \leqslant x < \xi, \quad (6.3a) \\ \dfrac{m-1}{m-2} \cdot B'_{m-1}(x), & x = \xi, \quad (6.3b) \\ (m-1)(x-\xi)^{m-2} \displaystyle\int_x^1 \frac{dB_{m-1}(\tau)}{(\tau-\xi)^m}, & \xi < x \leqslant 1. \quad (6.3c) \end{cases}$$

Proof. The first as well as the third line in (6.3) is easily proved using integration by parts in (6.2), while (6.3b) comes directly from differentiating equation (6.1). ∎

We are now interested in the following problem: Suppose we insert a new knot ξ into the set of knots of the B-spline B_{m-1} of order $m-1$, thus defining a B-spline B_m of order m. Evidently, the norm of B_m depends on the location of ξ; the problem is now to investigate this dependence. In the next theorem we first study the *location* of the new maximum, while in the final section 7 we shall investigate in a more general context the dependence of the *value* $\|B_m\|$ on ξ.

THEOREM 6. *Assume that the above-mentioned assumptions concerning the differentiability are all satisfied; furthermore, let x_m^* and x_{m-1}^* be such that*

$$B_m(x_m^*) = \|B_m\| \quad \textit{and} \quad B_{m-1}(x_{m-1}^*) = \|B_{m-1}\|.$$

Then the location of x_m^* depends on ξ in the following way:

$$\textit{If} \quad \xi > x_{m-1}^*, \quad \textit{then} \quad x_{m-1}^* < x_m^* < \xi, \tag{6.4a}$$

$$\textit{if} \quad \xi = x_{m-1}^*, \quad \textit{then} \quad x_{m-1}^* = x_m^* = \xi, \tag{6.4b}$$

$$\textit{if} \quad \xi < x_{m-1}^*, \quad \textit{then} \quad x_{m-1}^* > x_m^* > \xi. \tag{6.4c}$$

Proof. First let $\xi > x_{m-1}^*$; then in view of (6.3a) we have

$$B'_m(x_{m-1}^*) = (m-1)(\xi - x_{m-1}^*)^{m-2} \int_0^{x_{m-1}^*} \frac{dB_{m-1}(\tau)}{(\xi-\tau)^m} > 0,$$

hence $x_{m-1}^* < x_m^*$. Moreover, from (6.3b) it follows that $B'_m(\xi) < 0$. Thus $\xi > x_m^*$, and relation (6.4a) is proved.

Of course, the proof of (6.4c) is in analogy to that of (6.4a). Finally, relation (6.4b) follows directly from the fact that $B'_m(\xi) = 0$ if $\xi = x_{m-1}^*$, due to (6.3b). ∎

7. For Which Classes of B-Splines Does the Norm Tend to Zero?

Let there be given, for $m \in \mathbb{N}$, an infinite triangular matrix M of knots $x_\mu^{(m)}$, $\mu = 0, 1, ..., k_m$, which satisfy

$$0 = x_0^{(m)} < x_1^{(m)} < \cdots < x_{k_m - 1}^{(m)} < x_{k_m}^{(m)} = 1,$$

where to every knot $x_\mu^{(m)}$ a multiplicity $\nu_\mu^{(m)}$ is prescribed. (If there is no confusion possible, we shall sometimes drop the upper index (m).)

We consider to each row of M the corresponding B-spline $B_m(x \mid M)$, and analyze the question, for which knot matrices M the sequence of norms

$$\{\|B_m(\cdot \mid \mathrm{M})\|\}_{m=1}^{\infty}$$

tends to zero. Our first impression was that this is the case for "almost all" of them, i.e., for all B-splines except for the maximal ones given in section 4. But this is not true at all, as can be seen from the following two examples.

Example 1. Consider, for arbitrary $x_1 \in (0, 1)$ and $m \geqslant 3$, the B-spline $B_m(x \mid {}^{0}_{1}\, {}^{x_1}_{m-2}\, {}^{1}_{2})$, whose norm was in (4.7) computed to be

$$\left(\frac{m-2}{m-x_1-1}\right)^{m-2} = \left(1 - \frac{x_1 - 1}{m-2}\right)^{-(m-2)}.$$

Then

$$\lim_{m \to \infty} \left\| B_m\left(\cdot \,\middle|\, \begin{matrix} 0 & x_1 & 1 \\ 1 & m-2 & 2 \end{matrix}\right)\right\| = e^{x_1 - 1} > 0.$$

Note that B_m is a C^1-function for each finite value of m!

Example 2. Let $m \geqslant 2$ and $0 < \varepsilon < 1$. We consider the B-spline

$$B_m^*(x) := B_m\left(x \,\middle|\, \begin{matrix} 0 & \xi_1 & \cdots & \xi_{m-1} & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{matrix}\right)$$

with the simple knots

$$\xi_\mu := \frac{1 + \varepsilon^{2m-\mu}}{2}, \qquad \mu = 1, ..., m-1.$$

For $x \in [0, \xi_1)$, B_m^* takes according to (3.4), (3.6) the form

$$B_m^*(x) = \frac{(-1)^{m-1}}{\prod_{\mu=1}^{m-1} (-\xi_\mu)} \cdot x^{m-1} = \frac{(2x)^{m-1}}{\prod_{\mu=1}^{m-1} (1 + \varepsilon^{2\mu})}.$$

Since $\frac{1}{2} \in [0, \xi_1)$, it follows that for all m

$$\|B_m^*\| \geqslant B_m^*\left(\frac{1}{2}\right) = \frac{1}{\prod_{\mu=1}^{m-1} (1+\varepsilon^{2^\mu})} > \frac{1}{\prod_{\mu=1}^{\infty} (1+\varepsilon^{2^\mu})} = 1 - \varepsilon^2 > 0.$$

However, we do not like to close this paper with a series of negative examples, and so we will present in the following necessary and sufficient conditions, under which the B-spline's norms tend to zero as m goes to infinity. The first one, stated in Theorem 7, is of necessary type; it can be verified that the knot distributions in the above two examples do not satisfy this condition.

To formulate the theorem, we first need the B-splines' *expectation*

$$e_m := \int_{-\infty}^{\infty} x \cdot m \cdot B_m(x)\, dx \tag{7.1}$$

and *variance*

$$\sigma_m^2 := \int_{-\infty}^{\infty} m \cdot (x - e_m)^2 \cdot B_m(x)\, dx. \tag{7.2}$$

It should be noted that these expressions can easily be computed numerically by means of the formulae

$$e_m = \frac{1}{m+1} \cdot \sum_{\mu=0}^{k} v_\mu x_\mu \tag{7.3}$$

and

$$\sigma_m^2 = \frac{1}{(m+1)^2 (m+2)} \cdot \sum_{\substack{\mu, \tau = 0 \\ \mu < \tau}}^{k} v_\mu v_\tau (x_\mu - x_\tau)^2, \tag{7.4}$$

see e.g. [10, 14]. The above-mentioned necessary condition then reads as follows.

THEOREM 7. *If, for* $m \to \infty$,

$$\|B_m\| \to 0, \tag{7.5}$$

then

$$m \cdot \sigma_m \to \infty. \tag{7.6}$$

Proof. For all $\varrho > 0$, one has from (7.2) that

$$\sigma_m^2 \geqslant \int_{|x-e_m| \geqslant \varrho} m \cdot (x-e_m)^2 \cdot B_m(x)\, dx \geqslant m\varrho^2 \int_{|x-e_m| \geqslant \varrho} B_m(x)\, dx,$$

hence

$$\int_{|x-e_m| \geqslant \varrho} B_m(x)\, dx \leqslant \frac{\sigma_m^2}{m\varrho^2}. \tag{7.7}$$

Now, since $\int_{-\infty}^{\infty} B_m(x) = 1/m$, it follows from (7.7) that

$$2\varrho\, \|B_m\| \geqslant \int_{|x-e_m| < \varrho} B_m(x)\, dx \geqslant \frac{1}{m} \cdot \left(1 - \frac{\sigma_m^2}{\varrho^2}\right),$$

and therefore

$$\|B_m\| \geqslant \frac{1}{2\varrho m} \cdot \left(1 - \frac{\sigma_m^2}{\varrho^2}\right).$$

Substituting now $\varrho := \sqrt{3}\sigma_m$, we obtain

$$\|B_m\| \geqslant \frac{1}{3\sqrt{3}} \cdot \frac{1}{m\sigma_m},$$

which finally proves Theorem 7. ∎

It is conjectured that condition (7.6) or a slightly modified form of it may also be sufficient for (7.5) to hold; this will be a subject of future research.

We close this paper by presenting another type of sufficient condition, which says that the norm of a given B-spline *decreases*, if we move an arbitrary knot of this B-spline away from its maximum x_m^*. We shall see then in a corollary that this implies zero convergence for large classes of B-splines.

THEOREM 8. *Let there be given a continuously differentiable B-spline*

$$B_m(x) = B_m\left(x \,\middle|\, \begin{matrix} 0 & x_1 & \cdots & x_{k-1} & 1 \\ \nu_0 & \nu_1 & \cdots & \nu_{k-1} & \nu_k \end{matrix}\right)$$

of order m, and x_m^* such that $B_m(x_m^*) = \|B_m\|$.

Furthermore, consider another B-spline $\hat{B}_m(x)$ *of order* m, *which is constructed from* B_m *by shifting precisely one knot, say* x_j, *away from* x_m^*, *i.e.*

$$\hat{B}_m(x) = B_m\left(x \,\middle|\, \begin{matrix} 0 & x_1 & \cdots & x_j & x_j+\varepsilon & \cdots & x_{k-1} & 1 \\ \nu_1 & \nu_1 & \cdots & \nu_j - 1 & 1 & \cdots & \nu_{k-1} & \nu_k \end{matrix}\right) \tag{7.8}$$

with

$$\varepsilon \begin{cases} >0, & \text{if } x_j \geqslant x_m^*, \\ <0, & \text{if } x_j < x_m^* \end{cases} \tag{7.9}$$

such that $x_j + \varepsilon \in [0, 1]$. *Then*

$$\|\hat{B}_m\| < \|B_m\|.$$

Remark. In the second line of (7.9), we would write $x_j \leqslant x_m^*$ as well; the strict inequality sign was taken only for unicity of the definition.

Proof of Theorem 8. We make use of the following auxiliary result, which can be proved in analogy to Theorem 6, see also [11, Lemma 5.2]:

As in Section 6, let there be given a B-spline B_{m-1} whose maximum value is located at x_{m-1}^*. Now, assume that we insert alternatively two different new knots, say ξ and η, to this B-spline's set of knots, and denote the locations of the new maxima by

$$x_m^*(\xi) \quad \text{resp.} \quad x_m^*(\eta).$$

Then the following relations hold:

$$\text{If} \quad x_{m-1}^* \leqslant \xi < \eta, \quad \text{then} \quad x_m^*(\xi) < x_m^*(\eta), \tag{7.10a}$$

$$\text{if} \quad x_{m-1}^* \geqslant \xi > \eta, \quad \text{then} \quad x_m^*(\xi) > x_m^*(\eta), \tag{7.10b}$$

Now, assume without any loss of generality that $x_j \geqslant x_m^*$, and consider the particular B-spline

$$B_{m-1}(x) = B_{m-1}\left(x \,\middle|\, \begin{matrix} 0 & x_1 & \cdots & x_j & \cdots & x_{k-1} & 1 \\ \nu_1 & \nu_1 & \cdots & \nu_j - 1 & \cdots & \nu_{k-1} & \nu_k \end{matrix}\right)$$

of order $m-1$. Note that, if we insert the knot $\xi := x_j$, we recover B_m (and therefore $x_m^* = x_m^*(x_j)$), while inserting $\eta := x_j + \varepsilon$ gives us back $\hat{B}_m$.

Since $B_m'(x_m^*(x_j)) = 0$, relation (6.1) implies that

$$\|B_m\| \; (= B_m(x_m^*(x_j))) = B_{m-1}(x_m^*(x_j)) \tag{7.11}$$

and also

$$\|\hat{B}_m\| = B_{m-1}(x_m^*(x_j + \varepsilon)). \tag{7.12}$$

But we know from Theorem 6 that both $x_m^*(x_j)$ and $x_m^*(x_j+\varepsilon)$ lie in the interval $[x_{m-1}^*, 1]$. Since B_{m-1} is strictly decreasing in this interval, the combination of equations (7.10a), (7.11) and (7.12) proves the assertion of Theorem 8. ∎

COROLLARY. *Let there be given a sequence of B-splines $\{B_m\}$ such that the inner knots $x_1^{(m)}, ..., x_{m-1}^{(m)}$ satisfy*

$$x_\nu^{(m)} \leqslant \frac{\nu}{m} \quad \text{and} \quad x_{m-\nu}^{(m)} \geqslant \frac{m-\nu}{m} \quad \text{for} \quad \nu = \begin{cases} 1, ..., \dfrac{m-1}{2} & m \text{ odd}, \\ 1, ..., \dfrac{m}{2} & m \text{ even}. \end{cases}$$

Then

$$\mu_m \leqslant \|B_m\| \leqslant \beta_m,$$

with μ_m and β_m as in sections 4 and 5, i.e., there are two positive numbers c_1, c_2, such that

$$c_1 m^{-1/2} \leqslant \|B_m\| \leqslant c_2 m^{-1/2} \qquad \text{for} \quad m \to \infty$$

holds.

Proof. Follows directly from the combination of (4.8), (5.2) and Theorem 8. ∎

So we have finally seen that there is yet a quite big class of B-splines with zero convergence of the norms. For example, this is true for the well-known perfect splines [14, p. 139], which correspond to the simple knots

$$x_\nu^{(m)} = \sin^2\left(\frac{\nu\pi}{2m}\right).$$

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